Mathematics in physics lessons: developing structural skills

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Abstract

Many physics teachers complain that their students do not know enough mathematics. However, it seems that the domain of basic mathematics skills does not guarantee success in physics, once using mathematics in physics is much more complex than the straightforward application of rules and calculation. In fact, in spite of the deep interrelations between physics and mathematics, confirmed both by historical and epistemological studies, in the context of physics education, mathematics tend to be seen as a mere tool to quantify physical entities and express the relations between them. In order to face up to this inconsistency, we consider a distinction between technical skills – the ones related to the domain of basic rules of mathematics and normally developed in math's classes – and structural skills – which are related to the capacity of recognizing the structural role of mathematics in physical thought. We believe that one of the most important abilities to deal with phenomena in physics domain is to be able to use mathematics as a reasoning instrument. For that reason, we try to understand the physicist's use of this structural approach in a didactic context. University physics lessons given by a distinguished professor on Electromagnetism and Special Relativity were videotaped in order to investigate the structural approach in real classroom situations. The analysis of these lessons allowed us to identify a set of four structural skills which are defined and exemplified.

Introduction

The deep interdependency between mathematics and physics is well acknowledged by several exponents of both fields. Many physicists have stressed the indispensable role of mathematics in physics. Among them, Galilei (1854, pp. 60) wrote that "the universe is written in mathematical language" and Einstein (1934, pp. 117) believed that "the actual creative principle in physics lies in mathematics".

But mathematics also benefits from physics and this is recognized by many mathematicians. Kline (1960, pp. vii) stated that "mathematics is primarily man's finest creation for the investigation of nature", while Poincaré (1958, pp. 82) underlined "physics not only gives mathematicians the occasion to solve problems; it makes them foresee the solution; it suggests arguments to them".

In spite of this briefly outlined mutual interplay between mathematics and physics, it is possible to say that in the context of education, these two disciplines tend to be treated separately and the students hardly become aware of this successful interrelation. In physics classes, it is not uncommon to find mathematics being treated as a mere tool to quantify empirical relations and to solve standard problems by finding formulas in a "mathematical

toolbox”. In maths classes, physics tends to be seen as an application of previously defined mathematical abstract concepts.
We believe that there is a strong inconsistency between the historical interdependency and the educational independent approach of mathematics and physics which claims for a systematic research effort. Likewise, Hestenes (2003, pp. 104) states that “the challenge is to seriously consider the design and use of mathematics as an important subject for Physics Education Research”.
Aiming at contributing to this challenge, we propose a distinction between ***technical*** and ***structural skills*** (Pietrocola 2008) to analyze students’ and teachers’ use of mathematics in physics and the comprehension of their interdependence. The ***technical skills*** are related to the instrumental ability to apply mathematical rules and algorithms in physics, whereas the ***structural skills*** are associated to the use of mathematics to structure physical thought and to the recognition of the deep connection between the physical content and the mathematical formulation of a particular concept.
Our main goal is to characterize the latter ones and investigate how they can be approached in physics lessons. From the analysis of university physics lessons, we derived a set of four structural skills which are defined and exemplified.

Method

Concerning the methodological design of our research, we decided to conduct a case-study. Our data consist of the recordings from two university level physics courses performed by a particular professor at the University of São Paulo: 13 lessons on Special Relativity and 35 lessons on Electromagnetism, which makes a total of approximately 70 hours of video. We focused our attention on the moments where the professor used mathematical structures to explain basic concepts and to solve problems.
It is important to mention that the lessons from this professor were not chosen by chance. This particular professor was chosen due to the fact that he is widely admired and recognized, both by his students and colleagues, for having a “different approach”, since he normally encourages his students to reason about the physical interpretation of the mathematical formalism. The hypotheses we had was that his approach would focus on highlighting the structural role of mathematics, instead of its technical aspect.
The analysis of the data led us to identify a set of four structural skills. In the next section, we define each one of them and present examples of how they were approached during the lessons.

Structural skills

- *Mathematizing (from physics to mathematics)*

The first identified structural skill is called *mathematizing* and is related to the process of translating from the physical world (conceptions about nature, phenomenological observations and experimental data) to mathematics (mathematical structures and formulas). Being successful in this transfer process depends on being able to think mathematically, which involves not only a significant understanding of mathematical concepts and theories, but also the ability of abstracting, idealizing and modelling physical phenomena.
In fact, this very complex process is quite often taken for granted in physics lessons. We observed that the professor gave a special attention and dedicated a considerable amount of time to *mathematizing*. This approach normally took place in the introduction of a new idea or concept. Some sentences were extracted from the recordings and are commented below.

M1: This frame of reference is a reasoning instrument; it is not in nature, but in your mind.
M2: You make a mathematical cut in the wire, with your mind.

Here we realize that the professor is concerned on highlighting that mathematics is a human construct which is not found in nature. It is an abstraction created by the human ***mind***. This can sound obvious at a first glance, but from the professor's intention to mention it explicitly, it seems as if he wanted to stress that it is not. The ***mathematical cut*** from the second sentence is an attempt to make clear the distinction between a real and a mental cut (with an infinitesimally small length *dx*) of the wire.

M3: In order to maintain the symmetries of space and time this rule has to be a linear transformation.

This statement was extracted from the Special Relativity course during the derivation of the Lorentz transformations. It reflects the professor's intention of ***justifying*** the use of a particular mathematical structure (linear transformation) due to the physical desire of maintaining the symmetries of space and time. We believe that being able to identify the essential aspects that justify the use of mathematical structures in physical phenomena is a crucial ability for a meaningful mathematization. In trying to describe the process of translating from physics to mathematics, Redish (2005) states that:

> We map our physical structures into mathematical ones – create a mathematical model. To do this, **we have to understand what mathematical structures are available and what aspects of them are relevant to the physical characteristics we are trying to model** (Redish, 2005, pp. 7, our emphasis).

In general, we found that the development of the mathematizing skill was a conscious goal of the lessons, due to both the great amount of time dedicated and to the constant concern of mentioning it explicitly. However, whenever one mathematical expression/formula was reached, the professor tried to give it a physical meaning. That leads us to the second structural skill.

- *Interpreting (from mathematics to physics)*

After presenting or deriving a mathematical expression, either when introducing a new concept or solving a problem, the professor's focus was immediately directed to the physical interpretation of its meaning. We identified this skill as ***interpreting*** and noticed that it also played a central role in his lessons. Not a single time a formula was presented without an explanation of its physical meaning. This was done with the aid of both powerful schemas and an intensive use of gestures. We exemplify this approach with the sentences below.

I1: What does this expression mean? What does it say?
I2: It is important that you realize the power of this expression. It has the instruction for you to draw the arrow at any place in space.

We realize that the professor is claiming for a deep understanding of mathematical expressions. In the first sentence he asks his students to say what a particular equation means, as if it were possible to "read" it out loud. In the second, the power and meaning of the electric field's equation is explained.

I3: What do you physically expect from the result? Test/play with the result. What if x, y or z = 0? Is it reasonable? Does it make sense? If you don't expect anything from an equation your lost.
I4: If you go far away for the bar, shouldn't it be like a dot? Your result must be coherent.

These sentences show that identification of special/limit cases is an important strategy used by the professor to develop the skill of ***interpreting***. After reaching a particular expression, he usually encouraged his students to verify its consistency by comparing some results given by the expression with their previous physical expectations. His explicit remark that one should always expect something from an equation is mostly interesting.

I5: How did the information of the problem got into the calculation? […] The shape is given by the integration limits.
I6: I'm not asking you how to solve a double integral; I want you to know what you are doing when you integrate.

The meaningful understanding of the formalism is once again emphasized in these phrases. It becomes clear that the professor wants his students to make a conscious use of mathematics instead of applying it as a mere tool. When he says that he doesn't want them to solve, but to know what they are doing when solving a double integral, it is clear that he considers the structural role hierarchically superior to the technical.
The interplay between mathematizing and interpreting was a constant presence in his lessons. However, in some particular moments, a deeper discussion between the relation between mathematics and physics was conducted and some essential aspects of this interplay were explicitly mentioned. Therefore, we decided to classify these moments in two other categories, namely derivation and analogies, which are described and analyzed below.

- *Derivation (logic/deductive reasoning)*

The idea of proof is in the core of mathematical reasoning. This notion, which was mainly developed by the Greek philosophers, involves starting from an "evident" set of axioms and, by logical deductions, being able to prove a certain theorem. This style of reasoning is widely used and exemplified in Euclid's Elements and can also be found in several Physics' masterpieces, such as Newton's Principia and Einstein's work on Special Relativity. In fact, Einstein explicitly mentioned the similarity between geometry and theoretical physics by saying that:

> The theorist's method involves his using as his foundation of general postulates or principles from which he can **deduce** conclusions. His work thus falls into two parts. He must first discover his principles and then draw the **conclusions that follow from them** (Einstein, 1934, pp. 110, our emphasis).

Therefore, being able to fully understand logical derivations of formulas is an important ability which allows one to recognize how physical assumptions, such as the principles of minimal action or energy conservation, are "imposed" by physics during these derivations. Even though this approach was much less common during the lessons, we were also able to find moments where it was explicitly discussed.

D1: Can you demonstrate this equality from a mathematical point of view? The answer is no!

During a certain lesson of the Special Relativity course, the goal was to derive an expression which could transform the expressions of the electric and magnetic fields in a frame of reference to another which was moving uniformly relatively to the first. This was the answer given by the professor when one student demanded a mathematical reason for a particular equality during the derivation. The professor explained that the physical condition imposed is that the electromagnetism is covariant, i.e., that Maxwell's equations should have the same form in both frames. Therefore, the reason for assuming that two expressions should be equal comes from a physical principle and not from logical reasoning.

D2: We can show that this law (Gauss') is more general. We can derive Coulombs' law from Gauss'. We show where does 4π comes from.

The emphasis on the words ***show*** and ***derive*** are evidences that the deductive aspect of physics is being approached. The greater generality of Gauss' law is expressed when the professor mentions that it can be used to derive Coulombs' law and to show where the *mysterious* 4π comes from. Accordingly, Feyman (1965, pp. 26-3) states that "the real glory of science is that we can find a way of thinking such that the law is evident".
We strongly believe that derivations enhance students' knowledge about the origin of physics' equations, allow them to penetrate into the inner structure of physics' reasoning and avoid the rote memorization of senseless mathematical formulas. However, it should be conducted very carefully and consciously, justifying every step by mentioning each physical imposition, so that it doesn't become an artificial set of logical steps for the students.

- *Analogies (hidden similarities)*

Noticeably, one of the most fruitful resources of reasoning in physics is analogy, since the relation between the model and the modelled phenomenon is generally analogical. According to Hesse (1953, pp. 202), "an analogy in physics is a relation, either between two hypotheses, or between a hypothesis and certain experimental results, in which certain aspects of both relata can be **described by the same mathematical formalism**". In this sense, Steiner (1998, pp. 3) defends the idea that "the only way scientists found to arrive at the atomic and subatomic laws of nature was through mathematical analogies".
The discussion of formal similarities and the identification of unifying mathematical structures highlight the importance of analogical reasoning for physics' students and was also found in several moments of the lessons.

A1: Today we learned some strategies to deal with distributions of things. This is very general, it can be with charge, mass, population, anything that needs to be distributed.
A2: The mathematics of these two equations is the same.

At the end of the lesson on the mathematical description of charge distribution, the professor wanted to make clear that the strategy learned for that purpose was general and could be applied in other several cases. This conscious remark reflects his intention of catching students' attention for the role of analogical reasoning in physics.

A3: Is there a Gauss' law for gravitation? Yes. What is the flux of g through this mathematical surface? It is the Earth's mass. Which is the analogue to mass in Gauss' law? The charge!
A4: Every time you work in hydrodynamic – bees, water air – the flux is something that passes through a surface. They took this mathematical formulation to use in many other cases. But there are important differences. The electrical field doesn't really flow, it doesn't have any velocity.

Once again the intention of establishing connections and identifying hidden similarities becomes evident. In both cases, different physical contents are exposed to underline the formal similarities between them. However, it is important to notice that not only similarities but also differences are stressed, like in the case of the water flux having a velocity and electrical field not.

Preliminary conclusions and perspectives

Our main result so far is the set of four structural skills, which were defined and exemplified. However, it seems that there is a significant difference between them, since *mathematizing* and *interpreting* were found in almost every lesson, whereas *derivation* and *analogies* took place in crucial points. In order to better resolve these differences we intend to analyze the interplay between the categories using a time scale. We are interested in investigating how does the professor switch from one approach to another and how much time is dedicated to each skill along the whole course.

The presented paper is part of an ongoing PhD research in Physics Education. Due to the concrete examples and the identified categories concerning the structural approach of mathematics in physics lessons, the conducted case-study turned out to be an appropriate strategy.

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